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THE THREE-DIMENSIONAL MOTION OF OPTIMAL PYRAMIDAL BODIES†

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The three-dimensional inertial motion of pyramidal bodies, optimal in their depth of penetration, formed from parts of planes tangential to a circular cone and having a base in the form of a rhombus or a star, consisting of four symmetrical cycles, is investigated using the numerical solution of the Cauchy problem of the complete system of equations of motion of a body. It is assumed that the force action of the medium on the body can be described within the framework of a local model, when the pressure on the body surface can be represented by a two-term formula, quadratic in the velocity, and the friction is constant. It is shown that the stability criterion, obtained previously for the rectilinear motion of a pyramidal body on the assumption that the perturbed motion of the body is planar, also enables one, in the case of an arbitrary specification of the small perturbations of the parameters leading to the tree-dimensional motion of the body, to determine the nature of development of these perturbations. It is shown that if the rectilinear motion of the body is stable, its perturbed three-dimensional motion can be represented in the form of the superposition of plane motions, and when investigating each of them, the analytical solution of the plane problem obtained earlier can be used.

Exact solutions of problems of optimizing the shape of a body (with respect to the resistance and depth of penetration) within the framework of the model of local interaction were obtained in [1–4] for a specified length and a specified area of the base of the body without simplifying assumptions regarding its geometry. A method of constructing optimal shapes was proposed in [1–4], which enables an infinite set of optimal configurations to be constructed, all of which, for the same conditions of penetration into the medium, have the same resistance and ensure the same depth of penetration.

Shape optimization was carried out for rectilinear motion of the body in [1–4]. The motion of the body in the medium may be perturbed, and then, as the results of experimental and theoretical investigations show [5–9], the features of the body shape turn out to have an important influence on the development of the perturbations and nature of motion of the body. In the case of unstable motion, the velocity of the centre of mass of the body may differ considerably from its initial direction, and the trajectory of motion may have a curve form. An increase in the perturbations may lead to overturn of the body, in which case it is impossible to achieve the theoretically predicted depth of penetration. Hence, an investigation of the effect of perturbations on the characteristics of the motion of optimal shapes and a classification of these shapes with respect to stability of motion are important stages in investigating the properties of optimal bodies.

In the case of plane motion, within the framework of the model when the pressure on the body surface is represented by a two-term formula, quadratic in the velocity, while the friction is constant, such an investigation was carried out in [9] for pyramidal bodies, formed from sections of planes, tangential to a circular cone. It is well known [1–4] that if the cone has an optimum aperture angle, such configurations belong to the class of optimal configurations. For non-separating flow around bodies and small perturbations, imposed at the initial instant of time on the parameters of the rectilinear motion, an analytical solution of the problem of the plane motion of thin bodies, the contour of the base of which is a rhombus or a star, consisting of four symmetrical cycles, was constructed in [9]. A criterion of the stability of motion was obtained, which enables one, for known velocities, mass and positions of the centre of gravity of the body, to determine the nature of the perturbed motion of a pyramidal body.

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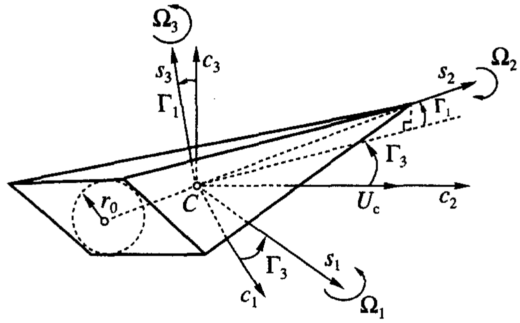

Fig. 1

In general, the motion of a body in a medium is three-dimensional, and the problem of the stability of a pyramidal body for a three-dimensional development of the perturbations, imposed at the initial instant of time on the parameters of the rectilinear motion, remains an urgent one. Below the complete system of equations of motion of the body is investigated using the local interaction model based on a numerical solution of the Cauchy problem.

1. MODEL OF THE FORCE ACTION ON THE BODY

Consider the three-dimensional inertial motion of a pyramidal body, made up of parts of planes, tangential to a circular cone, and having a base in the form of a rhombus or a star, consisting of four symmetrical cycles. An example of a rhombus-shaped configuration is shown in Fig. 1. A star-shaped body can be constructed from parts of surfaces of two rhombus-shaped forms, rotated by 90° with respect to one another around a longitudinal axis.

We will assume that, at the initial instant of time, the body is completely immersed in the medium and is not deformed during the motion. We will also assume that the effect of the free surface of the medium, as also the effect of gravity, on the motion of the body can be neglected.

We will write the force exerted by the medium on the body in the form

$$\mathbf{F} = \iint_S [\sigma_n \mathbf{n} + \sigma_\tau \boldsymbol{\tau}] dS \tag{1.1}$$

where σ_n and σ_τ are the normal and shear stresses on the body surface, $\mathbf{n}$ and τ, are the unit vectors of the inward normal and the normal tangential to the element of the surface, and integration is carried out over the contact area of the medium and the body S.

The interaction between the body and the medium, as in the case of plane motion [9], will be considered using the local model, assuming that each element of the surface S interacts with the medium independently of the other parts of the body. To write the normal stress we will use a two-term formula containing dynamic and stability terms, while the friction on the body surface will be assumed to be constant.

$$\sigma_n = A_1(\mathbf{U}\cdot\mathbf{n})^2 + C_1, \quad \sigma_\tau = C_2 \tag{1.2}$$

The positive coefficients A_1, C_1 and C_2 are constant parameters of the model, determined by the characteristics of the medium, $\mathbf{U}$ is the overall velocity of an element of the surface: $\mathbf{U} = \mathbf{U}_c + [\boldsymbol{\Omega} \times \mathbf{r}]$, where $\mathbf{U}_c$ is the velocity of the centre of mass of the body, $\boldsymbol{\Omega}$ is the angular velocity of rotation of the body and $\mathbf{r}$ is the radius vector of the element with origin at the centre of mass.

It was shown in [10] that for certain assumptions the stresses on the surface of the body when it moves in a gas and dense media such as soil and metals, are described by expressions (1.2). The term C_1 in this case represents the resistance of the medium to deformation, while the coefficient A_1 is of the order of the density of the medium. The model with constant friction (1.2) is often used when calculating the forces acting on a body penetrating into elastoplastic media (see, for example, [7, 8]), and then $C_2 = \tau_s$, where τ_s is the plastic friction. For specific media the values of A_1, C_1 and C_2 are taken either from the solution of model problems [11, 12], or are determined experimentally. For example, for clay media, according to the solution obtained for an incompressible elastoplastic medium in [12], we can assume

$$A_1 = 3\rho_0/2, \quad C_1 = 5\tau_s(1+\ln(\mu/\tau_s))/3, \quad C_2 = \tau_s \tag{1.3}$$

where ρ_0 is the density of the medium and μ is the shear modulus.

Within the framework of the local interaction model the vector $\boldsymbol{\tau}$ is coplanar with the vectors **U** and **n**

$$\boldsymbol{\tau} = [[\mathbf{U}\times\mathbf{n}]\times\mathbf{n}]/|[\mathbf{U}\times\mathbf{n}]| \tag{1.4}$$

while the surface S is defined by the condition

$$(\mathbf{u}\cdot\mathbf{n}) \le 0 \tag{1.5}$$

where **u** is the unit vector of the velocity of an element of the surface: $\mathbf{u} = \mathbf{U}/|\mathbf{U}|$.

2. THE PROPERTIES OF OPTIMAL PYRAMIDAL BODIES

The chosen model enables us to represent the force **F** (1.1) explicitly in terms of the shape parameters of the body and the parameters of the motion, in which the characteristics of the media occur as constants. This property is common to all the local models, and it was used to solve the problem of optimizing the shape of a body with respect to its resistance [1, 2, 4] and depth of penetration [3], when the resistance and depth of penetration can be represented in the form of functions which depend explicitly on the shape of the body.

For rectilinear motion of the body, which has a specified base area S_b, within the framework of the model (1.2)–(1.5) for the resistance of the body F_d, $F_d = -(\mathbf{F}\cdot\mathbf{s}_2)$, where $\mathbf{s}_2$ is the unit vector of the longitudinal axis of the body in the direction of motion (see Fig. 1), we can write the limit [1]

$$F_d \ge F^* = A_1 U_0^2 S_b f(\alpha^*) \tag{2.1}$$

Here U_0 is the velocity of motion of the body while α^* is the value of $\alpha = -(\mathbf{n}\cdot\mathbf{s}_2) \in [0, 1]$, for which the function of the real variable

$$f(\alpha) = \alpha^2(1+D_0); \quad D_0 = (C_1 + C_2(1-\alpha^2)^{1/2}/\alpha)/(A_1 U_0^2\alpha^2) \tag{2.2}$$

reaches a minimum. Bodies at each point of the surface of which $\alpha = \alpha^*$ have the minimum value of F_d, equal to F^*. For rectilinear motion $\mathbf{s}_2 = \mathbf{u}$ and, consequently, the shape of such bodies is formed by parts of surfaces, the normal of which makes a constant optimum angle with the direction of motion.

It was shown in [3] that one can use a similar method to form bodies which penetrate into the medium with initial velocity U_0 and which, for a specified mass m and a specified base area S_b, give maximum depth of penetration. Using the model (1.2)–(1.5) for bodies of maximum depth of penetration, the optimum value of α^* is found as the value $\alpha \in [0, 1]$, for which the function

$$h(\alpha) = \ln(1+1/D_0)/\alpha^2 \tag{2.3}$$

reaches a maximum.

The functions $f(\alpha)$ and $h(\alpha)$ are independent of the quantities S_b and m, and the values of α^* for them are determined by the characteristics of the medium and the initial velocity of motion U_0. In general, the values of α^* for the function $f(\alpha)$ and $h(\alpha)$ are different. For example, if for model (1.2) and the parameters (1.3) we consider a medium with $\rho_0 = 1500$ kg/m^3 and $\tau_s = 1$ MPa, and we assume $C_1 = 5\tau_s$, which corresponds in order of magnitude to the parameters of soil of moderate strength, then for $U_0 = 600$ m/sec the extrema of the functions $f(\alpha)$ and $h(\alpha)$ are reached for $\alpha^* = 0.085$ and $\alpha^* = 0.115$ respectively.

The simplest optimal body is a circular cone, half the aperture angle of which $\beta^* = \arcsin(\alpha^*)$. The method of constructing optimal shapes [1–4] enables us to construct an infinite set of optimal configurations, including pyramidal ones, made up of the parts of planes, tangential to the optimal cone. For example, the body shown in Fig. 1 belongs to the class of optimal bodies if the sides of the rhombus which form the contour of its base, touch circles of radius $r_0 = L\,\mathrm{tg}\,\beta^*$, where L is the length of the body.

For specified values of S_b and m, using the parameters of a pyramidal body, the expression for the maximum depth of penetration H^* [3] can be written in the form

$$H^* = h(\alpha^*)L/(2A_m), \quad A_m = 3A_1/\rho_m \tag{2.4}$$

where ρ_m is the mean density of the body: $\rho_m = m/V$, $V = LS_b/3$ is the volume of the body.

Solutions of problems of optimizing the body shape were obtained in [1–4] for rectilinear motion of the body. The motion of a body in a medium may be perturbed and the optimum values of F^* (2.1) and H^* (2.4) may then be reached only when the motion of the body is stable, and small perturbations, imposed at the initial instant of time on the parameters of the rectilinear motion, decay with time.

In the case of plane motion, assuming that the bodies are thin,

$$\beta^2 \ll 1, \quad \beta = (S_b/\pi)^{1/2}/L \tag{2.5}$$

and the medium does not separate from the body surface, a criterion of the stability of rectilinear motion of pyramidal bodies, the contour of the base of which is a rhombus or a star, consisting of four symmetrical cycles, was obtained in [9]. For values of D_0 (see the second expression of (2.2)) that are less than or of the order of unity, this criterion can be written in the following two equivalent forms

$$A_m > A_f, \quad A_f = (18/P_f)(C_m - 2(1 + \alpha^2/P_f)/3) \tag{2.6}$$

$$z_y = C_k - C_m > 0, \quad C_k = A_m P_f/18 + 2(1 + \alpha^2/P_f)/3 \tag{2.7}$$

Here z_y is the margin of stability of the body, C_m is the distance from the vertex of the body to its centre of mass and C_k is the distance from the vertex of the body to the critical position of the centre of mass for which loss of stability occurs. The parameter P_f is the shape parameter, where for star-shaped bodies $P_f = 1$ while for rhombus-shaped bodies $P_f = P$, where P is given by expression

$$\begin{aligned} P &= P_{1,2} = 1 \pm (1 - b^2)^{1/2} \\ b &= (R_1/R_k)^2; \quad R_1 = \alpha/\beta, \quad R_k = \pi^{1/2}/2 \approx 0.89 \end{aligned} \tag{2.8}$$

When $b < 1$ ($R_1 < R_k$), there are two values of P: $P_1 \in [1, 2)$ and $P_2 \in (0, 1]$, where $P_f = P_1$, if the centre of mass of the body moves in the plane of symmetry of the body, where the vertex of the rhombus with the smaller radius lies, and $P_f = P_2$, if the centre of mass of the body moves in the plane where the vertex of the rhombus with the larger radius lies. In the first case the rhombus shape has been called [9] horizontal, and in the second, vertical.

The value of A_f, which occurs in condition (2.6) in terms of the parameters α, β and C_m, depends on the shape of the body and on the position of the centre of mass. For pyramidal bodies with a uniform mass distribution over the volume $C_m = 3/4$. For $\alpha = \alpha^* = 0.115$, inequality (2.6) is the criterion of stability of the motion of the optimal body, which, for rectilinear motion and the conditions mentioned above, ensures maximum depth of penetration. For uniform bodies, optimal in the depth of penetration, the values of A_f are shown in Fig. 2 as a function of β for a star-shaped body (curve 1) and for a horizontal rhombus shape (curve 2) and a vertical rhombus shape (curve 3). All the curves emerge from the point A, which corresponds to the value $b = 1$, when the contour of the base is a square, and all bodies have the same geometry: $P_f = 1$.

For thin bodies (2.5) the value of β is half the aperture angle of the circular cone, equivalent to pyramidal bodies in length and area of the base. For a cone the parameter $P_f = 1$, and it was shown in [9] that if we put $\alpha = \beta$ in formulae (2.6) and (2.7), then inequalities (2.6) and (2.7) will define the condition for stable motion of the cone. The values of A_f corresponding to a uniform cone are shown in Fig. 2, curve 4.

The properties of the medium affect the stability of the motion of the body only in terms of the parameter A_1, which occurs in the expression for A_m (2.4). The values of A_m, calculated using formulae (1.3) and (2.4) for $\rho_0 = 1500$ kg/m^3, are shown in Fig. 2 along the ordinate axis by points, each of which corresponds to a uniform body made from the following materials: Al – aluminium ($A_m = 2.5$), Ti – titanium ($A_m = 1.5$), Fe – steel ($A_m = 0.85$) and W – tungsten ($A_m = 0.38$).

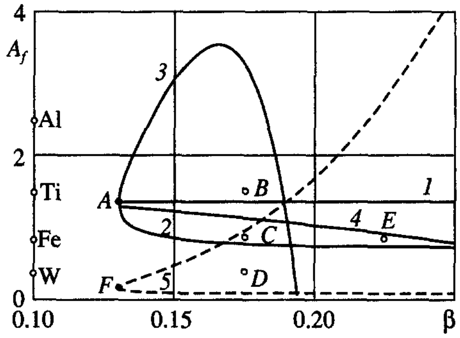

Fig. 2

It follows from an analysis of Fig. 2, in combination with condition (2.6), in particular, that for fixed C_m and α the form of the motion of star-shaped bodies (see curve 1) is independent of β: A_f = const ($A_f = 1.34$ when $C_m = 3/4$ and $\alpha = 0.115$), whereas the value of β has a considerable effect on the form of motion of a cone and rhombus-shaped bodies. It can be seen that when $A_m < 3.5$ there are values of β for which the motion of one of the rhombus-shaped bodies is stable, while the other is unstable.

In general, the perturbed motion of the body in the medium is three-dimensional, and the problem of the stability of the motion of a pyramidal body for a three-dimensional development of the perturbations, imposed at the initial instant of time on the parameters of rectilinear motion, is investigated below by a numerical solution of the Cauchy problem of the complete system of equations of motion.

3. THE EQUATIONS AND PARAMETERS OF THE THREE-DIMENSIONAL MOTION OF A BODY

The system of equations of the three-dimensional motion of a rigid body consists of the kinematic and dynamic equations of motion. The latter can be represented in vector form in terms of the derivatives with respect to time of the momentum and angular momentum vectors of the body (see, for example [3]):

$$m d\mathbf{U}_c/dt = \mathbf{F}, \quad d\mathbf{K}/dt = \mathbf{M} \tag{3.1}$$

where $\mathbf{K} = J\boldsymbol{\Omega}$, J is the inertia tensor of the body, and $\mathbf{M}$ is the moment of the forces acting on the body which, like the force $\mathbf{F}$, is calculated using model (1.2)–(1.5).

The scalar form of writing Eqs (3.1) depends on the choice of the system coordinates and the set of kinematic parameters, defining the motion of the body. When investigating the three-dimensional dynamics of pyramidal bodies, according to traditions in aerodynamics [13], four right Cartesian systems of coordinates were used, the origins of which at the initial instant time coincided with the centre of mass of the body C: fixed, trajectory, velocity and coupled. It was assumed that the unit vectors $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$ of the coupled system of coordinates $Cs_1s_2s_3$ are directed along the principal axes of inertia of the body, and the Cs_2 axis is directed along its longitudinal axis in the direction of motion of the body (see Fig. 1).

For unperturbed motion of the body, the directions of the corresponding axes of all four systems of coordinates coincide, and the vector of the velocity of the centre of mass $\mathbf{U}_c$ is directed along the OY axis of the fixed system of coordinates $OXYZ$.

The perturbed motion of the body, as is assumed in aerodynamics [13], will be characterized by the following set of kinematic parameters: the angles of attack Γ_1, velocity of roll Γ_2 and slip Γ_3, and angular velocity of pitch Ω_1 roll Ω_2 and yaw Ω_3. The angles of attack Γ_1 and slip Γ_3 specify the orientation of the axes of the velocity system of coordinates $Cc_1c_2c_3$ with respect to the coupled system (Fig. 1), and the nature of the development of the perturbations with respect to these angles defines the type of motion of the body. The angle of the velocity of roll Γ_2 specifies the angle of rotation of the trajectory system of coordinates with respect to the Cc_2 axis, for which the trajectory system converts into the velocity system.

For the kinematic parameters Γ_i and Ω_i ($i = 1, 2, 3$) coupling equations exist (see [13]), which, together with the kinematic equation of translational motion of the body: $d\mathbf{R}/dt = \mathbf{U}_c$, where $\mathbf{R}$ is the vector of the position of the centre of mass of the body relative to a fixed system of coordinates, and the dynamic equations (3.1) form a closed system of equations. The system consists of twelve equations, and, for an arbitrary specification of the initial values of the parameters, this system can only be integrated numerically.

A program for the numerical solution of the Cauchy problem of the complete system of the equations of motion was written based on a fourth-order Runge–Kutta method, and calculations obtained using it are employed below to investigate the features of the three-dimensional motion of a pyramidal body.

4. FEATURES OF THE THREE-DIMENSIONAL MOTION OF PYRAMIDAL BODIES

Without loss of generality we will assume that, at the initial instant of time, the positions of the three systems of coordinates: fixed, trajectory and velocity, coincide, and the vector of the velocity of the centre of mass $\mathbf{U}_c$ is directed along the OY axis.

Numerical calculations, the results of which are discussed below, were carried out for a medium with the parameters given in Section 2, with an initial velocity of motion of the body $U_0 = 600$ m/s. In this case, a maximum of the function $h(\alpha)$ (2.3) is reached when $\alpha = 0.115$, and pyramidal bodies, optimal in the depth of penetration, were constructed from parts of planes tangential to a cone with aperture angle 13°. The results of a calculation for such bodies with a uniform mass distribution over the volume are given below.

When describing the results of the solution of the problem we will use dimensionless quantities γ_i and ω_i: $\gamma_i = \Gamma_i/\beta$ and $\omega_i = \Omega_i L/(U_0\beta)$, where $i = 1, 2, 3$, and we will assume that all the linear dimensions refer to the body length L. The plane motion of the body, the stability of which for non-separating penetration is defined by condition (2.6) and (2.7), is described by a set of parameters $\gamma_i = \omega_i = 0$ for non-zero values of one of the pairs: (γ_1, ω_1) or (γ_3, ω_3).

Numerical simulation of the three-dimensional motion of the body was carried out for arbitrary initial values of γ_i and ω_i. However, when investigating the stability of the rectilinear motion of the optimal pyramidal body and in order to compare the results of the investigation with the analytical results obtained earlier in [9], the initial perturbations with respect to γ_i and ω_i were taken to be such that at the initial instant of time there was no zone of the medium separation from the body surface.

When carrying out the calculations particular attention was devoted to the region of the parameters of the pyramidal body in which the stability of the plane motion of a rhombus-shaped body depended on the plane of motion. Thus, for example, the points B and C in Fig. 2, the ordinates of which correspond to the values of A_m for uniform bodies, made from titanium and steel, lie in this region in the plane of the parameters (β, A_m). According to stability criterion (2.6), the motion of a horizontal rhombus-shaped body (see curve 2 in Fig. 2) with parameters β and A_m, corresponding to the points B and C, is stable, while the motion of a vertical rhombus (see curve 3) is not stable. Note that the point B lies above, while the point C lies below curves 1 and 4, constructed for values of A_f for uniform star-shaped bodies (curve 1) and circular cones (curve 4), equivalent to pyramidal bodies in mass, length and base area. According to condition (2.6), the plane motion o these bodies is stable in the first case and unstable in the second. Also, according to condition (2.6), we obtain that the motion of all the pyramidal configurations of tungsten with parameters β and A_m denoted by the point D in Fig. 2, is unstable, where with parameters corresponding to the point E (a configuration of steel), the motion of rhombus-shaped bodies is stable, while the motion of a cone and a star-shaped body is unstable.

For values of β and A_m corresponding to the points B, C, D and E in Fig. 2, we carried out calculations which model the three-dimensional motion of the bodies. The initial perturbations with respect to the slip angle were taken to be three times greater than for the angle of attack: $\gamma_1 = 0$ and $\gamma_3 = 0.3$, while the perturbations with respect to the angular velocity were as follows: $\omega_1 = 0.1$, $\omega_2 = 0$ and $\omega_3 = 0.3$.

The trajectories of motion of the centre of mass of the bodies obtained as a result of the calculation are shown in Figs 3–5 in projections onto the plane of the fixed system of coordinates $OXYZ$ in the form of sets I, II and III of continuous curves B, C, D and E. The sets were constructed for the trajectories of the centre of mass of star-shaped bodies (set I), a cone (set II) and rhombus-shaped bodies (set III) for values of the parameters β and A_m corresponding to the points B, C, D and E and Fig. 2.

The types of three-dimensional motions of star-shaped bodies and cones corresponded exactly to the types of motion described above for these bodies in agreement with the criterion of stability of plane motion (2.6). For unstable motion, the initial perturbations with respect to γ_i and ω_i increased with time, which led to a deviation of the vector $\mathbf{U}_c$ from its initial direction and bending of the trajectory of motion of the centre of mass (see curves C, D, and E for sets I and II). Note that the margin of stability z_y (2.7) of a star-shaped body is less than the margin of stability of the equivalent cone, while the trajectories of the centre of mass of star-shaped bodies bend more strongly than the trajectories of the cone. An increase in the perturbations with respect to γ_i and ω_i for tungsten configurations led to their inversion (see curves D), where the star-shaped body overturned earlier than the cone. The bodies themselves

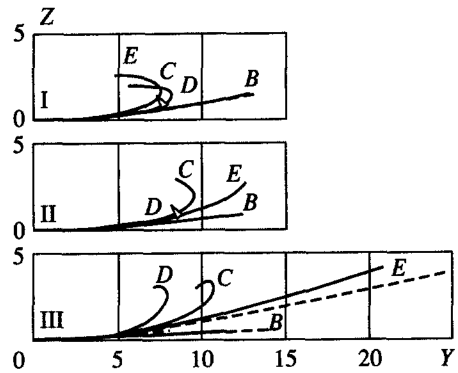

Fig. 3

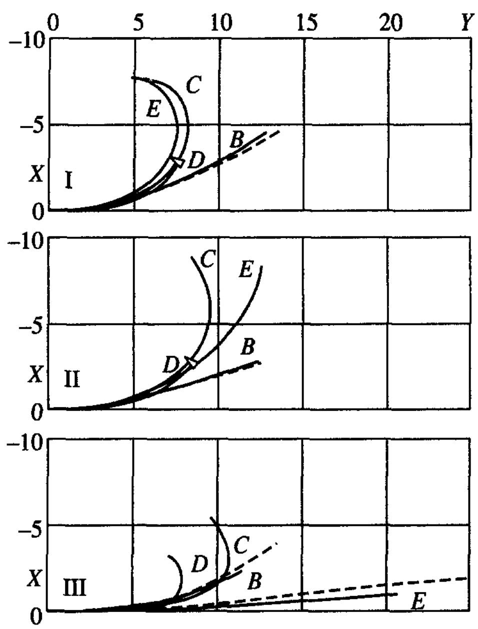

Fig. 4

are shown schematically in Figs 3 and 4 at the instant of overturn. For stable motion, the perturbations with respect to γ_i and ω_i attenuated with time, and the angle of deviation of the vector $\mathbf{U}_c$ from the initial direction reached the asymptote defining a new direction of rectilinear motion of the body. For plane motion, the direction of asymptote can be found from the analytical solution [9].

We compared the results obtained with the results of the plane motion of a body, for which we took as the initial values (γ_0, ω_0) for the analytical solution [9] the values of the pairs $(\gamma_1, \omega_1) = (0.1, 0.1)$ and $(\gamma_3, \omega_3) = (0.3, 0.3)$. In this case $(\gamma_0, \omega_0) = (\gamma_1, \omega_1)$ when constructing the analytical trajectory in the *OYZ* plane (see the dashed curves *B* in Fig. 3), and $(\gamma_0, \omega_0) = (\gamma_3, \omega_3)$ when constructing the trajectory in the *OXY* plane (see the dashed curves *B* in Fig. 4). It follows from a comparison of the results of the numerical and analytical solution solution that, for stable motion, the three-dimensional motion of the centre of mass of the body can be represented in the form of the superposition of plane motions, each of which is described by the analytical solution of the plane problem obtained earlier.

The projections of the trajectories of the three-dimensional motion of the centre of mass of optimal rhombus-shaped bodies are shown in Figs 3–5 in the form of sets of curves III. According to the criterion of stability of plane motion (2.6) only the trajectories *E* in Figs 3–5 correspond to the motion of a rhombus-shaped body, stable for plane motion in both planes of symmetry. The trajectories *B*, *C*, and *D* are constructed for the cases when the motion of the body is unstable in one or both planes of symmetry. It can be seen that for unstable modes of motion the use of optimal rhombus-shaped bodies

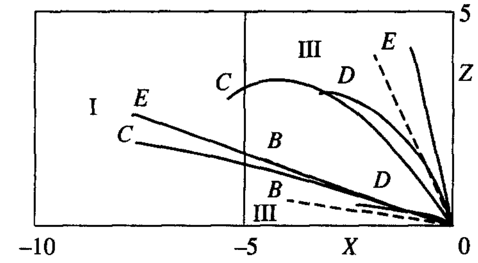

Fig. 5

does not have the same advantages as cones, which were obtained in [3] for rectilinear motion. However, when choosing the optimal body with parameters from the stable region of motion, these advantages become obvious (see curves E of sets II and III).

As follows from the analysis of the results of numerical solution, a characteristic feature of the motion of conical and star-shaped bodies is the fact that, irrespective of the types of motion, the trajectories of their centres of mass lie in practically the same plane, the inclination of which to the fixed system of coordinates $OXYZ$ is determined by the initial values of γ_i and ω_i (see the set of curves I in Fig. 5). Note that the inclination of the plane can be obtained if we use the analytical solution [9] (see the dashed curve B of set I in Fig. 5) using the rule described above. For a rhombus-shaped body this is not the case, and in the general case of unstable motion, the trajectories of its centre of mass are essentially three-dimensional (see curves B, C and D of set III in Fig. 5). However, for stable motion, the trajectory of a rhombus-shaped body, like cones and star-shaped bodies, is close to planar (see curve E of set III). When finding the projections of the trajectory onto the plane of the fixed system of coordinates $OXYZ$ one can use the analytical solution of the plane problem obtained earlier in [9], the results of which are given in Figs 3–5 by the dashed curves. The somewhat longer trajectory, obtained from the analytical solution, can be explained by the fact that when calculating its length we ignored the perturbations of the angular velocities, and the length of the trajectory was equated to the value of H^* (2.4), which is reached in the case of the rectilinear motion of the body.

When comparing the optimal star-shaped and rhombus-shaped bodies with respect to the margin of stability z_y (2.7), it should be borne in mind that in general the margin of stability of pyramidal bodies depends on the shape, mass and position of the centre of gravity of the body (the parameters α, β, A_m and C_m). However, the margin of stability of star-shaped bodies is independent of β, and it is always less than the margin of stability of the equivalent cone. The margin of stability of rhombus-shaped bodies depends very much on β, and for fixed values of α and C_m one can obtain a region of the parameters β and A_m where the margins of stability of both rhombus-shaped bodies will be greater than the margin of stability of star-shaped body. For values of A_m from this region, using relations (2.7) and (2.8), we can write the limit

$$12\alpha^2/P_1 \leq A_m \leq 12\alpha^2/P_2 \qquad (4.1)$$

where P_1 and P_2 are the shape parameters of the rhombus-shaped body (2.8).

For optimal bodies, the region of values of A_m (4.1) is situated to the right of curve 5 in Fig. 2, which is two branches emerging from the point F. As a result, we can assume that for values of β and A_m from this region, when choosing the configuration for an optimal body, preference should be given to a rhombus shape. The results of a numerical solution of the problem of the three-dimensional dynamics of a body agrees with this assumption (see curves C, D and E in sets I and III in Figs 3 and 4).

Numerical simulation of the three-dimensional motion of a body, the results of which have been discussed above, was carried out when there are no initial roll perturbations: $\omega_2(0) = 0$. However, when the body moves along a trajectory these perturbations appear and affect the characteristics of the motion of the body. In Fig. 6 curves 1–3 give the values of the parameters ω_i (the value of i corresponds to the number of the curve) as a function of the traversed length l for stable motion (the continuous curves) and unstable motion (the dashed curves) of steel rhombus-shaped bodies, which correspond to the points E and C in the plane of the parameters β and A_m in Fig. 2. Note that, for rectilinear motion, these configurations for the same length ensure the same depth of penetration H^* (2.4). However, a characteristic feature of the unstable three-dimensional motion of a rhombus-shaped body was a considerable development of the roll perturbations (see the dashed curve 2 in Fig. 6) and, as a consequence,

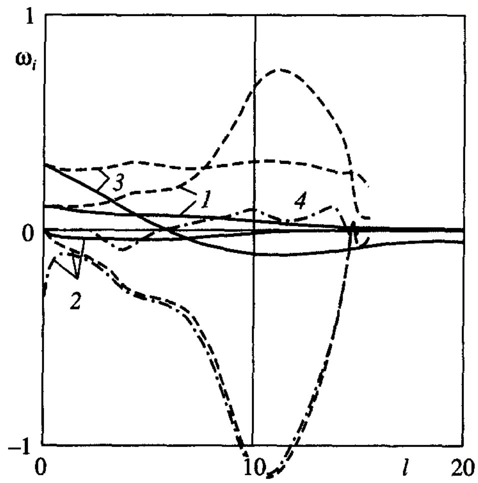

Fig. 6

their effect on the increase in the pitch and yaw perturbations. For stable motion, the perturbations which appear decay rapidly and in practice have no effect on the characteristics of motion of the body. As a result, for stable motion, the trajectory of the centre of mass of the body deviated much less from its initial direction, and its length was greater than for unstable motion by ~30% (see curves C and E in sets III in Figs 3 and 4).

The development of roll perturbations for star-shaped bodies has a different form, and for a steel star-shaped body with parameters corresponding to point C in Fig. 2, the change in the value of ω_2 is shown in Fig. 6 by the dash-dot curve 4. Characteristic features of the motion of star-shaped bodies where the slow development of roll perturbations and the change in the sign of ω_2 during the motion. These features were the reason why the trajectories of star-shaped bodies, like those of cones, were close to planar, and why the stability criterion obtained for these bodies in the case plane motions remains true for the three-dimensional development of perturbations. Note that a similar result was obtained in [8] for thin solids of revolution for non separating flow and when there is a small zone of the medium separation on their surfaces, and it was shown that the stability criterion obtained for these bodies for plane motion [7] remains true for the three-dimensional case also.

Analysis of the results of numerical calculations, carried out when there are initial roll perturbations, when $\omega_2(0) \neq 0$, showed that these perturbations do not introduce appreciable changes into the general pattern of the motion of a pyramidal body, considered above. The value of ω_2 in this case rapidly reaches the values which it has on the same part of the path when $\omega_2(0) = 0$ and, as an example, the change in the value of ω_2 as a function of the path traversed for an initial value of $\omega_2(0) = 0.3$ for rhombus-shaped bodies with parameters corresponding to the point C in Fig. 2, is shown by the dash–dot curve 2 in Fig. 6.

5. CONCLUSION

As a result of our investigations using model (1.2)–(1.5), we have derived the characteristic features of the three-dimensional motion of optimal pyramidal bodies, formed from parts of planes tangential to a circular cone and having a base in the form of a rhombus or a star, consisting of four symmetrical cycles. It has been shown that the stability criterion obtained previously in [9] for the plane motion of pyramidal bodies, enables one, in the case of an arbitrary specification of small perturbations of the parameters of the rectilinear motion, to determine the nature of their development. For stable motion of a pyramidal body when there is a three-dimensional development of the perturbations, it is necessary that the stability criterion should be satisfied for it in all planes of symmetry. According to the stability criterion, regions of parameters have been determined in which, when the configuration of the optimal body is chosen, one must give preference to a star-shaped or rhombus-shaped body. It has been shown that if the rectilinear motion of the body is stable, roll perturbations have no appreciable influence on the characteristics of the motion of the body, and its perturbed three-dimensional motion can be represented in the form of the superposition of plane motions, each of which is described by the analytical solution of the plane problem obtained previously in [9]. It has been confirmed that for unstable motion, the use of optimal bodies does not give the advantages which are obtained with them for rectilinear motion. However, for optimal pyramidal bodies with parameters from the region of stable motion, these advantages are close to those obtained when the body shape is optimized.

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